

Some graphs related to Thompson's group F

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Abstract

The Schreier graphs of Thompson's group F with respect to the stabilizer of $\frac{1}{2}$ and generators x_0 and x_1 , and of its unitary representation in $L_2([0, 1])$ induced by the standard action on the interval $[0, 1]$ are explicitly described. The coamenability of the stabilizers of any finite set of dyadic rational numbers is established. The induced subgraph of the right Cayley graph of the positive monoid of F containing all the vertices of the form x_nv , where $n \geq 0$ and v is any word over the alphabet $\{x_0, x_1\}$, is constructed. It is proved that the latter graph is non-amenable.

Introduction

Thompson's group F was discovered by Richard Thompson in 1965. A lot of fascinating properties of this group were discovered later on, many of which are surveyed nicely in [CFP96]. It is a finitely presented torsion free group. One of the most intriguing open questions about this group is whether F is amenable. Originally this question was asked by Geoghegan in 1979 (see p.549 of [GS87]) and since then dozens of papers were in some extent devoted to it. It was shown in [BS85] that F does not contain a nonabelian free subgroup and in [CFP96] that it is not elementary amenable. So the question of amenability of F is particularly important because F would be an example of a group given by a balanced presentation (two generators and

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two relators) of either amenable, but not elementary amenable group (the first finitely presented example was constructed by R. Grigorchuk in [Gri98]), or non-amenable group, which does not contain a non-abelian free subgroup (the first finitely presented example of this type was constructed by Ol'shanskii and Sapir in [OS02]).

The study of the Schreier graphs of F was also partially inspired by the question of amenability of F . In particular, if any Schreier graph with respect to any subgroup is non-amenable the whole group F would be non-amenable. Unfortunately, all Schreier graphs we describe here are amenable which does not give any information about the amenability of F . But the knowledge about the structure of Schreier graphs provides some additional information about F itself.

It happens that the described Schreier graph of the action of F on the set of dyadic rational numbers on the interval $(0, 1)$ is closely related to the unitary representation of F in the space $B(L_2([0, 1]))$ of all bounded linear operators on $L_2([0, 1])$. It reflects (modulo a finite part) the dynamics of F on the Haar wavelet basis in $L_2([0, 1])$. We define the Schreier graph of the group action on the Hilbert space with respect to some basis and make this connection precise.

R. Grigorchuk and S. Stepin in [GS98] reduced the question of amenability of F to the right amenability of the positive monoid P of F . Moreover, the amenability of F is equivalent to the amenability of the induced subgraph Γ_P of the Cayley graph Γ_F of F with respect to generating set $\{x_0, x_1\}$ containing the positive monoid P . We construct the induced subgraph Γ_S of Γ_F containing all the vertices of the form x_nv for $n \geq 0, v \in \{x_0, x_1\}^*$ and prove that this graph is non-amenable. In this construction we use the realization of the elements of the positive monoid of F as binary rooted forests. The existence of this representation was originally noted by K. Brown and developed by J. Belk in [Bel04] and Z. Šunić in [Šun07]. It was also used by J. Donnelly in [Don07] to construct an equivalent condition for amenability of F .

The structure of the paper is as follows. In Section 1 the definition and the basic facts about Thompson's group are given. Section 2 contains the description of the Schreier graph of the action of F on the set of dyadic rational numbers from the interval $(0, 1)$. The coamenability of the stabilizers of any finite set of dyadic rational numbers is shown in Section 3. The Schreier graph of the action of F on $L_2([0, 1])$ is constructed in Section 4. The last Section 5 contains a description of the subgraph Γ_S of Γ_P and the proof that Γ_S is non-amenable.

The author expresses warm gratitude to Rostislav Grigorchuk for valuable comments and bringing his attention to Thompson's group, and to Zoran Šunić, who has pointed to the connection with forest diagrams, which simplified the proofs in the last section.

1 Thompson's group

Definition 1. *The Thompson's group F is the group of all strictly increasing piecewise linear homeomorphisms from the closed unit interval $[0, 1]$ to itself that are differentiable everywhere except at finitely many dyadic rational numbers and such that on the intervals of differentiability the derivatives are integer powers of 2. The group operation is superposition of homeomorphisms.*

Basic facts about this group can be found in the survey paper [CFP96]. In particular, it is proved that F is generated by two homeomorphisms x_0 and x_1 given by

$$x_0(t) = \begin{cases} t, & 0 \leq t \leq \frac{1}{2}, \\ t - \frac{1}{4}, & \frac{1}{2} \leq t \leq \frac{3}{4}, \\ 2t - 1, & \frac{3}{4} \leq t \leq 1, \end{cases} \quad x_1(t) = \begin{cases} t, & 0 \leq t \leq \frac{1}{2}, \\ \frac{t}{2} + \frac{1}{4}, & \frac{1}{2} \leq t \leq \frac{3}{4}, \\ t - \frac{1}{8}, & \frac{3}{4} \leq t \leq \frac{7}{8}, \\ 2t - 1, & \frac{7}{8} \leq t \leq 1. \end{cases}$$

The graphs of x_0 and x_1 are displayed in Figure 1.

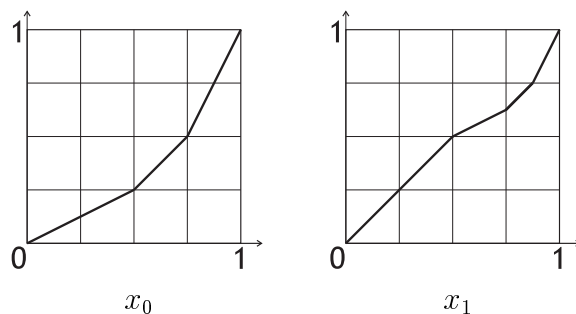


Figure 1: Generators of F

Throughout the paper we will follow the following conventions. For any two elements f, g of F and any $x \in [0, 1]$

$$(fg)(x) = g(f(x)), \quad f^g = gfg^{-1}. \quad (1)$$

With respect to the generating set $\{x_0, x_1\}$ F is finitely presented. But for some applications it is more convenient to consider an infinite generating set $\{x_0, x_1, x_2, \dots\}$, where

$$x_n = (x_1)^{x_0^{n-1}}.$$

With respect to this generating set (and with respect to convention (1)) F has a nice presentation

$$F \cong \langle x_0, x_1, x_2, \dots \mid x_k x_n = x_{n+1} x_k, \ 0 \leq k < n \rangle. \quad (2)$$

2 The Schreier graph of the action of F on the set of dyadic rational numbers

Let G be a group generated by a finite generating set S acting on the set M . The *Schreier graph* $\Gamma(G, S, M)$ of the action of G on M with respect to the generating set S is an oriented labelled graph defined as follows. The set of vertices of $\Gamma(G, S, M)$ is M and there is an arrow from $x \in M$ to $y \in M$ labelled by $s \in S$ if and only if $x^s = y$.

For any subgroup H of G , the group G acts on the right cosets in G/H by right multiplication. The corresponding Schreier graph $\Gamma(G, S, G/H)$ is denoted as $\Gamma(G, S, H)$ or just $\Gamma(G, H)$ if the generating set is clear from the context.

Conversely, if G acts on M transitively, then $\Gamma(G, S, M)$ is canonically isomorphic to $\Gamma(G, S, \text{Stab}_G(x))$ for any $x \in M$, where the vertex $y \in M$ in $\Gamma(G, S, M)$ corresponds to the coset from $G/\text{Stab}_G(x)$ consisting of all elements of G that move x to y .

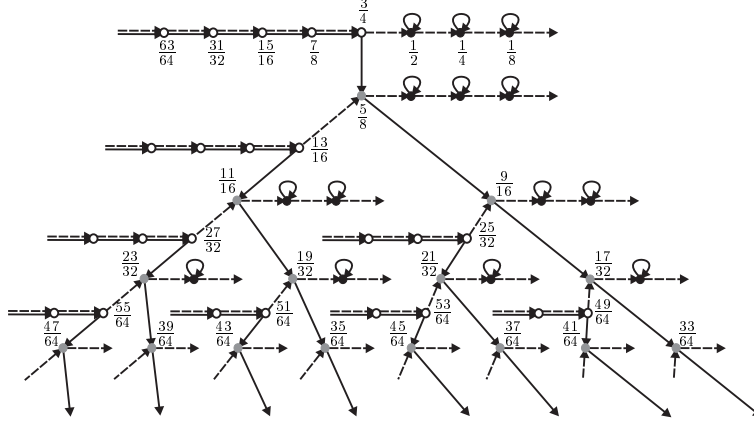
Consider the subgroup $\text{Stab}_F(\frac{1}{2})$ of F consisting of all elements of F that fix $\frac{1}{2}$. There is a natural isomorphism $\psi : \text{Stab}_F(\frac{1}{2}) \rightarrow F \times F$ given by

$$\text{Stab}_F\left(\frac{1}{2}\right) \ni f(t) \xrightarrow{\psi} \left(2f\left(\frac{t}{2}\right), 2\left(f\left(\frac{t+1}{2}\right) - 1\right)\right) \in F \times F. \quad (3)$$

This group was studied in [Bur99], where it was shown that it embeds into F quasi-isometrically.

The Schreier graph $\Gamma(F, \{x_0, x_1\}, \text{Stab}_F(\frac{1}{2}))$ coincides with the Schreier graph of the action of F on the orbit of $\frac{1}{2}$. Let D be the set of all dyadic rational numbers from the interval $(0, 1)$. It is known that F acts transitively on D (which follows also from the next proposition). Therefore the latter graph coincides with the Schreier graph $\Gamma(F, \{x_0, x_1\}, D)$.

Proposition 1. *The Schreier graph $\Gamma(F, \{x_0, x_1\}, D)$ has the following structure (dashed arrows are labelled by x_0 and solid arrows by x_1)*



Proof. Define the following subsets of D .

$$A_n = \left\{ \frac{k}{2^n} \mid k \text{ is odd} \right\} \cap \left(\frac{1}{2}, \frac{3}{4} \right), n \geq 3$$

$$B_n = \left\{ \frac{k}{2^n} \mid k \text{ is odd} \right\} \cap \left(\frac{3}{4}, \frac{7}{8} \right), n \geq 4$$

$$C_n = A_n \cap \left(\frac{1}{2}, \frac{5}{8} \right), \quad D_n = A_n \cap \left(\frac{5}{8}, \frac{3}{4} \right), n \geq 4$$

On the graph above, A_n represents the $(n - 3)$ -rd level of the gray vertices in the binary tree; B_n is the set of the white vertices between levels $n - 4$ and $n - 3$ of the tree, adjacent to 2 gray vertices; C_n and D_n are the sets of the gray vertices of the $(n - 3)$ -rd level having gray and white neighbors above respectively.

Now we compute the action of F on this subsets (see Figure 2). We have $x_0^{-1}(A_n) = B_{n+1}$, $x_1(B_n) = D_n$, $x_1(A_n) = C_{n+1}$, hence $(x_0^{-1}x_1)(A_n) = D_{n+1}$ and $(x_0^{-1}x_1)(A_n) \cup x_1(A_n) = A_{n+1}$. Furthermore, for any set $A \subset \mathbb{R}$ denote $\alpha A + \beta = \{\alpha a + \beta : a \in A\}$. Then $x_0^k(A_n) = x_0^k x_1(A_n) = 2^{-k+1}(A_n - \frac{1}{4})$ for $k \geq 1$. This corresponds to the rays with the black vertices sticking out to the right from the gray ones. On the other hand since the actions of x_0^{-1} and x_1^{-1} on $[\frac{3}{4}, 1]$ coincide, for any element f of length $k \geq 0$ from the monoid generated by x_0^{-1} and x_1^{-1} we have $f(B_n) = 1 - 2^{-k}(1 - B_n)$. This corresponds to the rays with white vertices. There is one more geodesic line in the graph corresponding to $\frac{1}{2}$ which completes the picture. \square

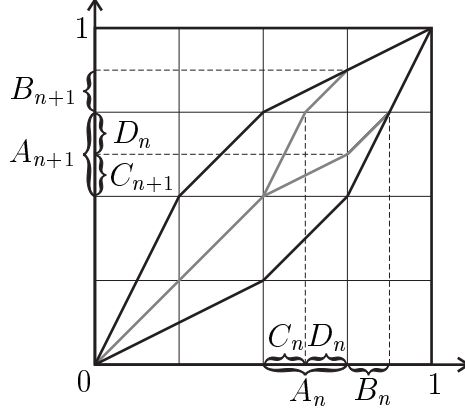


Figure 2: Dynamics of x_0 and x_1

This graph gives alternative proofs of the following well-known facts.

Corollary 1. *The subsemigroup of F generated by x_1 and $x_0^{-1}x_1$ is free.*

Corollary 2. (a) *Thompson's group F acts transitively on the set D of all dyadic rationals from the interval $(0, 1)$.*

(b) *$\text{Stab}_F(\frac{1}{2})$ acts transitively on the sets of dyadic rationals from the intervals $(0, \frac{1}{2})$ and $(\frac{1}{2}, 1)$.*

Proof. Part (a) follows immediately from the structure of the Schreier graph $F/\text{Stab}_F(\frac{1}{2})$. Part (b) is a consequence of part (a) and the isomorphism (3). \square

Proposition 2. *The subgroup $\text{Stab}_F(\frac{1}{2})$ is a maximal subgroup in F .*

Proof. Let f be any element from $F \setminus \text{Stab}_F(\frac{1}{2})$. Then for any $g \in F$ we show that $g \in \langle \text{Stab}_F(\frac{1}{2}), f \rangle$. Let g be an arbitrary element in F that does not stabilize $\frac{1}{2}$.

Denote $u = f(\frac{1}{2})$ and $v = g(\frac{1}{2})$. Without loss of generality we may assume $u < \frac{1}{2}$. Then by transitivity from Corollary 2(b) there exists $h \in \text{Stab}_F(\frac{1}{2})$ such that either $h(f(\frac{1}{2})) = v$ or $h(f^{-1}(\frac{1}{2})) = v$ depending on whether $v < \frac{1}{2}$ or $v > \frac{1}{2}$. In any case the element $\tilde{f} = fh$ (or $\tilde{f} = f^{-1}h$) belongs to $\langle \text{Stab}_F(\frac{1}{2}), f \rangle$ and satisfies $\tilde{f}(\frac{1}{2}) = v$.

Now for $\tilde{h} = g\tilde{f}^{-1}$ we have $\tilde{h}(\frac{1}{2}) = \tilde{f}^{-1}(g(\frac{1}{2})) = \tilde{f}^{-1}(v) = \frac{1}{2}$. Thus $\tilde{h} \in \text{Stab}_F(\frac{1}{2})$ and $g = \tilde{h}\tilde{f} \in \langle \text{Stab}_F(\frac{1}{2}), f \rangle$. \square

Proposition 1 also yields a bound on the length of an element. Namely, if the graph of an element $f \in F$ passes through the point (a, b) for some dyadic rational numbers a and b , then the length of f with respect to the generating set $\{x_0, x_1\}$ is not smaller than the combinatorial distance between a and b in the graph $\Gamma(F, \{x_0, x_1\}, D)$.

Estimates similar in spirit (also based on the properties of graph of an element, but in a different realization of F) were used by J.Burillo in [Bur99] to show that $\text{Stab}_F(\frac{1}{2})$ quasi-isometrically embeds into F .

3 Coamenability of stabilizers of several dyadic rationals

In this section we show that for any finite subset $\{d_1, \dots, d_n\}$ of dyadic rationals the Schreier graphs of F with respect to $\text{Stab}_F(d_1, \dots, d_n)$ is amenable, which, unfortunately, does not give any information about amenability of F .

First we recall the definition of an amenable graph.

Definition 2. *Given an infinite graph $\Gamma = (V, E)$ of bounded degree the Cheeger constant $h(\Gamma)$ is defined as follows*

$$h(\Gamma) = \inf_S \frac{|\partial S|}{|S|},$$

where S runs over all nonempty finite subsets of V , and ∂S , the boundary of S , consists of all vertices of $V \setminus S$ that have a neighbor in S .

Definition 3. *The graph Γ is called amenable if $h(\Gamma) = 0$.*

Definition 4. *A subgroup H of a group G is called coamenable in G if the Schreier graph $\Gamma(G, H)$ is amenable.*

Note, that coamenability of a subgroup does not depend on the generating set of G . This follows easily from Gromov's doubling condition (see Theorem A in Section 5).

Proposition 3. *Let $\{d_1, \dots, d_n\} \subset D$ be any finite subset of dyadic rationals. Then the subgroup $\text{Stab}_F(d_1, \dots, d_n)$ of F consisting of all elements stabilizing all the d_i 's is coamenable in F .*

Proof. First, we describe the structure of the Schreier graph $\Gamma(F, \{x_0, x_1\}, \text{Stab}_F(d_1, \dots, d_n))$, $d_1 < d_2 < \dots < d_n$. Analogously to the singleton case there is a one-to-one correspondence between cosets from $F/\text{Stab}_F(d_1, \dots, d_n)$ and all strictly increasing n -tuples of dyadic rationals. This follows from the fact that F acts transitively on the latter set (see [CFP96]). There is an edge labelled by $s \in \{x_0, x_1\}$ from the coset (d'_1, \dots, d'_n) to the coset (d''_1, \dots, d''_n) if and only if $s(d'_i) = d''_i$ for every i .

Geometrically one can interpret this in the following way. Consider a disjoint union of n copies of $\Gamma(F, \{x_0, x_1\}, \text{Stab}_F(\frac{1}{2}))$ (a layer for each d_i). Then the coset (d'_1, \dots, d'_n) of $F/\text{Stab}_F(d_1, \dots, d_n)$ can be represented by the path joining d'_i vertex on the i -th layer with d'_{i+1} vertex on the $(i+1)$ -th layer (see Figure 3). The action of the generators on the set of such paths is induced by the independent actions of the generators on the layers.

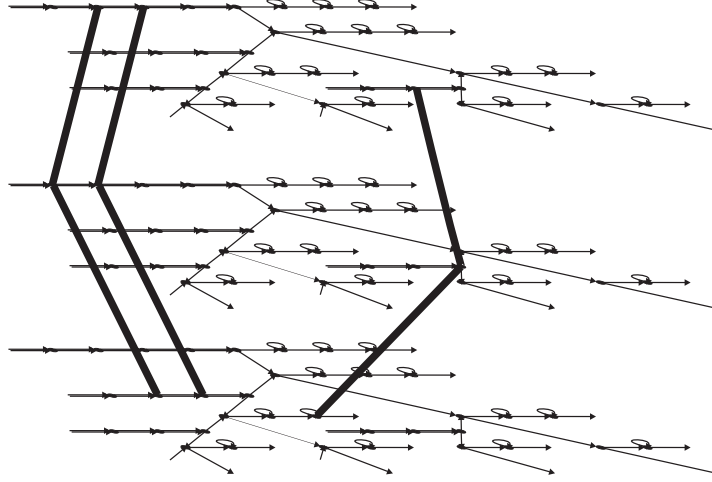


Figure 3: Cosets in $F/\text{Stab}_F(d_1, d_2, d_3)$

Now define

$$E_i = \left(\frac{1}{2^{i+n}}, \frac{1}{2^{i+n-1}}, \dots, \frac{1}{2^{i+1}} \right) \in F/\text{Stab}_F(d_1, \dots, d_n)$$

and

$$S_m = \{E_i \mid 1 \leq i \leq m\}.$$

Since $x_1(E_i) = E_i$ and $x_0(E_i) = E_{i+1}$ we have that the boundary $\partial S_m = \{E_0, E_{m+1}\}$ and

$$\lim_{m \rightarrow \infty} \frac{|\partial S_m|}{|S_m|} = \lim_{m \rightarrow \infty} \frac{2}{m} = 0.$$

Thus $h(\Gamma(F, \{x_0, x_1\}, \text{Stab}_F(d_1, \dots, d_n))) = 0$ and $\text{Stab}_F(d_1, \dots, d_n)$ is coamenable in F . \square

The amenability of the action of F on the set of dyadic rational numbers and on the set of the ordered tuples of dyadic rational numbers was also noted independently by N. Monod and Y. Glasner (private communication).

4 The Schreier graph of the action of F on $L_2([0, 1])$

There is a natural unitary representation of Thompson's group F in the space $\mathcal{B}(L_2([0, 1]))$ of all bounded linear operators on $L_2([0, 1])$. For $g \in F$ and $f \in L_2([0, 1])$ define

$$(\pi_g f)(x) = \sqrt{\frac{dg(x)}{dx}} f(g^{-1}x).$$

For our purposes it is convenient to consider this action with respect to the orthonormal Haar wavelet basis $B = \{h^{(0)}, h_j^{(i)}, i \geq 0, j = 1 \dots 2^i\}$ in $L_2([0, 1])$, where $h^{(0)}(x) \equiv 1$ and

$$h^{(0)}(x) \equiv 1, \quad h_1^{(0)}(x) = \begin{cases} -1, & x < \frac{1}{2}, \\ 1, & x \geq \frac{1}{2}, \end{cases}$$

$$h_j^{(i)}(x) = \begin{cases} -2^{\frac{i}{2}}, & \frac{j-1}{2^i} \leq x < \frac{j-1}{2^i} + \frac{1}{2^{i+1}}, \\ 2^{\frac{i}{2}}, & \frac{j-1}{2^i} + \frac{1}{2^{i+1}} \leq x < \frac{j}{2^i}, \\ 0, & x \notin [\frac{j-1}{2^i}, \frac{j}{2^i}]. \end{cases}$$

This basis has first appeared in 1910 in the paper of Haar [Haa10] and plays an important role in the wavelet theory (see, for example, [Dau92, WS01]).

The convenience of using this basis for us comes from the following fact. Each of the generators x_0 and x_1 acts on each of the basis

functions $h_j^{(i)}$ for $i \geq 3$ linearly on the support of $h_j^{(i)}$, so that the image also belongs to B . More precisely, straightforward computations yield

$$\begin{aligned}
\pi_{x_0} h_j^{(i)} &= h_j^{(i+1)}, \quad i \geq 1, \quad 1 \leq j \leq 2^{i-1}, \\
\pi_{x_0} h_j^{(i)} &= h_{j-2^{i-2}}^{(i)}, \quad i \geq 2, \quad 2^{i-1} + 1 \leq j \leq 2^{i-1} + 2^{i-2}, \\
\pi_{x_0} h_j^{(i)} &= h_{j-2^{i-1}}^{(i-1)}, \quad i \geq 2, \quad 2^{i-1} + 2^{i-2} + 1 \leq j \leq 2^i, \\
\pi_{x_1} h_j^{(i)} &= h_j^{(i)}, \quad i \geq 1, \quad 1 \leq j \leq 2^{i-1}, \\
\pi_{x_1} h_j^{(i)} &= h_{j+2^{i-1}}^{(i+1)}, \quad i \geq 2, \quad 2^{i-1} + 1 \leq j \leq 2^{i-1} + 2^{i-2}, \\
\pi_{x_1} h_j^{(i)} &= h_{j-2^{i-3}}^{(i)}, \quad i \geq 3, \\
&\quad 2^{i-1} + 2^{i-2} + 1 \leq j \leq 2^{i-1} + 2^{i-2} + 2^{i-3}, \\
\pi_{x_1} h_j^{(i)} &= h_{j-2^{i-1}}^{(i-1)}, \quad i \geq 3, \quad 2^{i-1} + 2^{i-2} + 2^{i-3} + 1 \leq j \leq 2^i.
\end{aligned} \tag{4}$$

There is a one-to-one correspondence ψ between $B \setminus \{h^{(0)}\}$ and the set of all dyadic rationals from the interval $(0, 1)$ given by $\psi(h_j^{(i)}) = \frac{j-1}{2^i} + \frac{1}{2^{i+1}}$, that is, each basis function corresponds to the point of its biggest jump (where the function changes the sign).

Below we will use the following simple observation, which can also be used to derive equalities (4). If a function $h(x) \in L_2([0, 1])$ changes its sign at the point x_0 then for any $g \in F$ the function $(\pi_g h)(x)$ changes its sign at the point $g(x_0)$. This enables us to find the image of $h_j^{(i)}$, $i \geq 3$ under action of π_{x_k} , $k = 0, 1$ in the following easy way:

$$\pi_{x_k} h_j^{(i)} = \psi^{-1}(x_k(\psi(h_j^{(i)})))$$

In other words the following diagram is commutative for $k = 0, 1$

$$\begin{array}{ccc}
h_j^{(i)} & \xrightarrow{\pi_{x_k}} & h_{j'}^{(i')} \\
\psi \downarrow & & \psi \downarrow \\
\frac{j-1}{2^i} + \frac{1}{2^{i+1}} & \xrightarrow{x_k} & \frac{j'-1}{2^{i'}} + \frac{1}{2^{i'+1}}
\end{array}$$

Now we define the Schreier graph of the action of a group on a Hilbert space.

Let \mathcal{H} be a Hilbert space with an orthonormal basis $\{h_i, i \geq 1\}$. Suppose there is a representation π of a group $G = \langle S \rangle$ in the space of all bounded linear operators $\mathcal{B}(\mathcal{H})$. We denote the image of $g \in G$ under π as π_g .

Definition 5. *The Schreier graph Γ of the action of a group G on a Hilbert space H with respect to the basis $\{h_i, i \geq 1\}$ of H and generating set $S \subset G$ is an oriented labelled graph defined as follows. The set of vertices of Γ is the basis $\{h_i, i \geq 1\}$ and there is an arrow from h_i to h_j with label $s \in S$ if and only if $\langle \pi_s(h_i), h_j \rangle \neq 0$ (in other words the coefficient of $\pi_s(h_i)$ at h_j in the basis $\{h_i, i \geq 1\}$ is nonzero).*

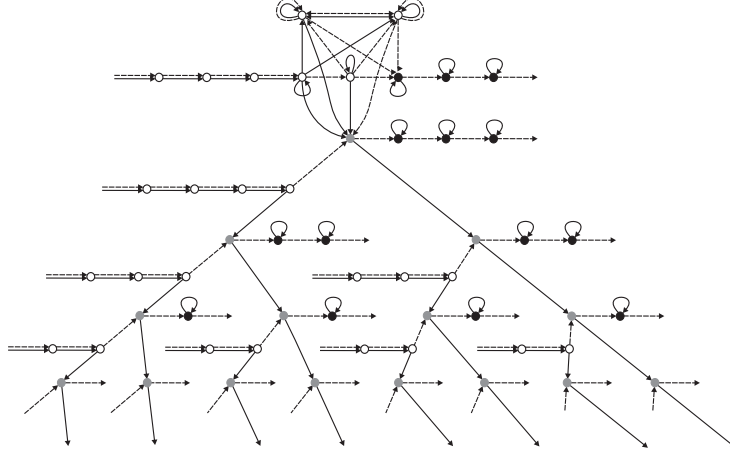
The argument above shows that the Schreier graph of the Thompson's group action on $L_2([0, 1])$ with respect to the Haar basis and generating set $\{x_0, x_1\}$ coincides modulo a finite part with the Schreier graph $\Gamma(F, \{x_0, x_1\}, D)$. In order to complete the picture we have to find the images under the action of π_{x_0} and π_{x_1} of those $h_j^{(i)}$ which are not listed in (4).

Again straightforward computations give the following equalities.

$$\begin{aligned} \pi_{x_0} h^{(0)} &= \left(\frac{1}{4} + \frac{\sqrt{2}}{2}\right) h^{(0)} - \frac{1}{4} h_1^{(0)} + \left(-\frac{1}{2} + \frac{\sqrt{2}}{4}\right) h_1^{(1)}, \\ \pi_{x_0} h_1^{(0)} &= \frac{1}{4} h^{(0)} + \left(-\frac{1}{4} + \frac{\sqrt{2}}{2}\right) h_1^{(0)} + \left(\frac{1}{2} + \frac{\sqrt{2}}{4}\right) h_1^{(1)}, \\ \pi_{x_0} h_2^{(1)} &= \left(\frac{1}{2} - \frac{\sqrt{2}}{4}\right) h^{(0)} + \left(\frac{1}{2} + \frac{\sqrt{2}}{4}\right) h_1^{(0)} - \frac{1}{2} h_1^{(1)}, \\ \pi_{x_1} h^{(0)} &= \left(\frac{5}{8} + \frac{\sqrt{2}}{4}\right) h^{(0)} + \left(-\frac{3}{8} + \frac{\sqrt{2}}{4}\right) h_1^{(0)} - \frac{\sqrt{2}}{8} h_2^{(1)} + \left(\frac{1}{4} - \frac{\sqrt{2}}{4}\right) h_3^{(2)}, \\ \pi_{x_1} h_1^{(0)} &= \left(-\frac{3}{8} + \frac{\sqrt{2}}{4}\right) h^{(0)} + \left(\frac{5}{8} + \frac{\sqrt{2}}{4}\right) h_1^{(0)} - \frac{\sqrt{2}}{8} h_2^{(1)} + \left(\frac{1}{4} - \frac{\sqrt{2}}{4}\right) h_3^{(2)}, \\ \pi_{x_1} h_1^{(1)} &= \frac{\sqrt{2}}{8} h^{(0)} + \frac{\sqrt{2}}{8} h_1^{(0)} + \left(-\frac{1}{4} + \frac{\sqrt{2}}{2}\right) h_2^{(1)} + \left(\frac{1}{2} + \frac{\sqrt{2}}{4}\right) h_3^{(2)}, \\ \pi_{x_1} h_4^{(2)} &= \left(-\frac{1}{4} + \frac{\sqrt{2}}{4}\right) h^{(0)} + \left(-\frac{1}{4} + \frac{\sqrt{2}}{4}\right) h_1^{(0)} + \left(\frac{1}{2} + \frac{\sqrt{2}}{4}\right) h_2^{(1)} - \frac{1}{2} h_3^{(2)}. \end{aligned}$$

These computations together with Proposition 1 prove the following proposition.

Proposition 4. *The Schreier graph of Thompson's group action on $L_2([0, 1])$ with respect to the Haar basis and the generating set $\{x_0, x_1\}$ has the following structure (dashed arrows are labelled by x_0 and solid arrows by x_1)*



5 Parts of the Cayley graph of F

Recall, that the positive monoid P of F is the monoid generated by all generators x_i , $i \geq 0$. As a monoid it has a presentation

$$P \cong \langle x_0, x_1, x_2, \dots \mid x_k x_n = x_{n+1} x_k, \ 0 \leq k < n \rangle,$$

which coincides with the infinite presentation (2) of F . The group F itself can be defined as a group of left fractions of P (i.e. $F = P^{-1} \cdot P$).

It was shown in [GS98] (see also [Gri90]) that the amenability of F is equivalent to the right amenability (with respect to our convention (1)) of P . Moreover, let Γ_F be the Cayley graph of F with respect to the generating set $\{x_0, x_1\}$ and Γ_P be the induced subgraph of Γ_F containing positive monoid P . The following proposition is of a folklore type.

Proposition 5. *Amenability of F is equivalent to amenability of the graph Γ_P .*

Proof. Any finite set T in F can be shifted to the positive monoid P , i.e. there is some $g \in F$ such that $Tg \subset P$. The boundary $\partial_P(Tg)$ of this shifted set in Γ_P is not bigger than the boundary of T in Γ_F . Hence, Cheeger constant of Γ_P is not bigger than the one of Γ_F . Thus, non-amenability of Γ_P implies non-amenability of F .

Suppose that Γ_P is amenable. Then for any $\varepsilon > 0$ there exists a subset T of P , such that its boundary $\partial_P T$ in Γ_P satisfies

$$\frac{|\partial_P T|}{|T|} < \frac{\varepsilon}{4} \quad (5)$$

Now we can bound the size of the boundary $\partial_F T$ of T in Γ_F . We use simple observations that for finite sets A and B of the same cardinality $|A \setminus B| = |B \setminus A| = \frac{1}{2}|A \Delta B|$ and that $|Tx_i^{-1} \Delta T| = |(Tx_i^{-1} \Delta T)x_i| = |Tx_i \Delta T|$.

We have

$$\partial_F T = (Tx_0 \setminus T) \cup (Tx_1 \setminus T) \cup (Tx_0^{-1} \setminus T) \cup (Tx_1^{-1} \setminus T).$$

Therefore,

$$\begin{aligned} |\partial_F T| &\leq |Tx_0 \setminus T| + |Tx_1 \setminus T| + |Tx_0^{-1} \setminus T| + |Tx_1^{-1} \setminus T| \\ &\leq \frac{1}{2}(|Tx_0 \Delta T| + |Tx_1 \Delta T| + |Tx_0^{-1} \Delta T| + |Tx_1^{-1} \Delta T|) \\ &= |Tx_0 \Delta T| + |Tx_1 \Delta T| = 2|Tx_0 \setminus T| + 2|Tx_1 \setminus T| \leq 4|\partial_P T| < \varepsilon|T| \end{aligned}$$

since $Tx_i \setminus T \subset \partial_P T$ for $i = 1, 2$ and by (5). This shows that Γ_F is also amenable in this case. \square

In this section we explicitly construct the induced subgraph Γ_S of Γ_F containing the set of vertices

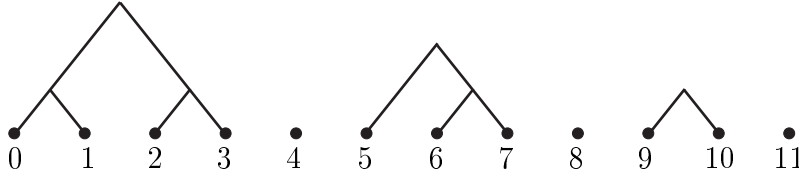
$$S = \{x_n u \mid n \geq 0, u \text{ is a word over } \{x_0, x_1\}\}. \quad (6)$$

We also prove that this graph is non-amenable.

Since S is included in the positive monoid of F and contains elements from the infinite generating set $\{x_0, x_1, x_2, \dots\}$, it is natural to use the language of forest diagrams developed in [Bel04, Šun07] (though the existence of this representation was originally noted by K.Brown [Bro87]). First we recall the definition and basic facts about this representation of the elements of F .

There is a one-to-one correspondence between the elements of the positive monoid of F and rooted binary forests. More generally, there is a one-to-one correspondence between elements of F and, so-called, reduced forest diagrams, but for our purposes (and for simplicity) it is enough to consider only the elements of the positive monoid.

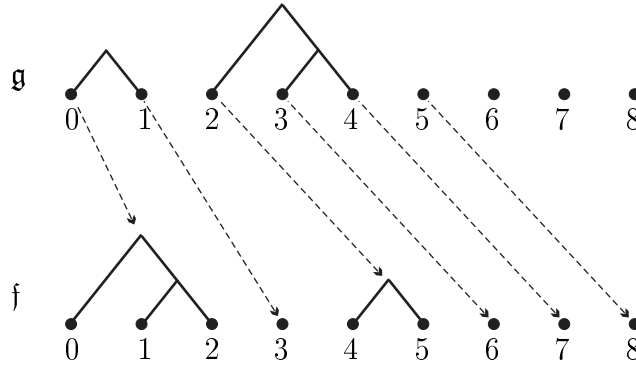
A *binary forest* is an ordered sequence of finite rooted binary trees (some of which may be trivial). The forest is called *bounded* if it contains only finitely many nontrivial trees.



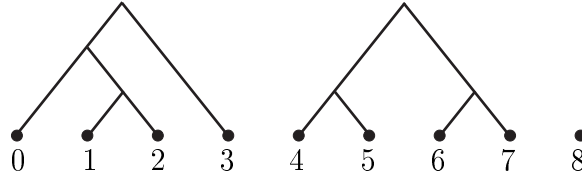
There is a natural way to enumerate the leaves of the trees in the forest from left to right. First we enumerate the leaves of the first tree from left to right, then the leaves of the second tree, etc. Also there is a natural left-to-right order on the set of the roots of the trees in the forest.

The product \mathbf{fg} of two rooted binary forests \mathbf{f} and \mathbf{g} is obtained by stacking the forest \mathbf{g} on the top of \mathbf{f} in such a way, that the i -th leaf of \mathbf{g} is attached to the i -th root of \mathbf{f} .

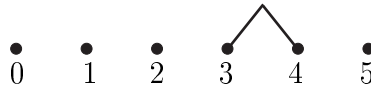
For example, if \mathbf{g} and \mathbf{f} have the following diagrams



then their product \mathbf{fg} is the following rooted binary forest



With this operation the set of all rooted binary forests is isomorphic (see [Bel04, Šun07]) to the positive monoid of Thompson's group F , where x_n corresponds to the forest in which all the trees except the $(n + 1)$ -st one (which has number n) are trivial and the $(n + 1)$ -st tree represents a single caret. Below is the picture of the forest corresponding to x_3 .



The multiplication rule for the forests implies the following algorithm for construction of the rooted forest corresponding to the element $x_{i_1}x_{i_2}x_{i_3}\cdots x_{i_n}$ of the positive monoid of F . Start from the

trivial forest (where all the trees are singletons) and consequently add the carets at the positions i_1, i_2, \dots, i_n (counting from 0 the roots of the trees in the forest in previous iteration).

For our main result in this section we need two lemmas.

Lemma 1. *Let u be a word from the positive monoid of the form $u = x_n v$, where $n \geq 2$ and v is a word over the alphabet $\{x_0, x_1\}$ of length at most $n - 2$. Then this word is not equal in F to any other word of the form $x_m w$, where w is a word over $\{x_0, x_1\}$.*

Proof. The forest diagram corresponding to u has a caret c connecting the n -th and $(n + 1)$ -st leaves corresponding to x_n and possibly some nontrivial trees to the left of c .

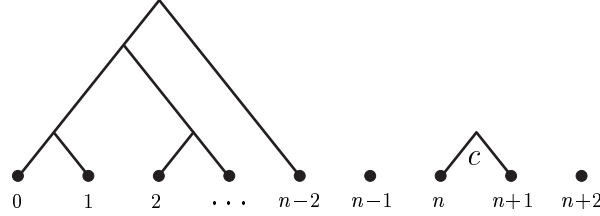


Figure 4: Forest corresponding to $x_n v$

Indeed, after attaching the caret corresponding to x_n all the other carets are attached at positions either 0 or 1. Each of these carets decreases the number of trees to the left of caret c by 1. Since originally there were n trees to the left from c and the length of v is at most $n - 2$, there must be at least 2 trees to the left of c in the forest representing u .

Suppose there is another word of the form $x_m w$ in the positive monoid of F whose corresponding rooted forest coincides with the forest of u . Since there are at least 2 trees to the left of caret c one can not obtain this caret by applying x_0 or x_1 . Therefore it was constructed at the first step with application of x_m . Thus $x_m = x_n$ because this caret connects the n -th and $(n + 1)$ -st leaves, which, in turn, implies that $v = w$ in F . But both v and w are the elements of a free submonoid generated by x_0 and x_1 , yielding that $x_n v = x_m w$ as words. \square

Lemma 2. *Let u be a word from the positive monoid of the form $u = x_n v x_1 v'$, where $n \geq 2$ and v is a word over the alphabet $X = \{x_0, x_1\}$*

of length $n - 2$. Then this word is not equal in F to any other word of the form $x_m w$, where w is a word over $\{x_0, x_1\}$.

Proof. The rooted forest corresponding to $x_n v$ is constructed in Lemma 1 and shown in Figure 4. Note, that there are exactly 2 trees (one of which is shown trivial in Figure 4) to the left of caret c . At the next step we apply generator x_1 , which attaches the new caret d that connects the root of the second of these trees to the root of caret c . The resulting forest is shown in Figure 5.

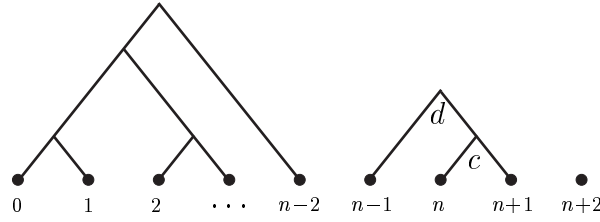


Figure 5: Forest corresponding to $x_n v x_1$

Next, applying v' adds some extra carets on top of the picture. The final rooted forest is shown in Figure 6.

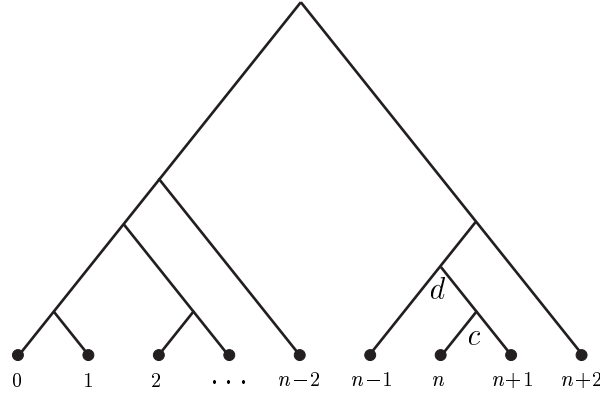


Figure 6: Forest corresponding to $x_n v x_1 v'$

Analogously to Lemma 1 we obtain that if the rooted forest of $x_m w$ coincides with the one of u , the caret c could appear only from the initial application of x_m (since it must be placed before caret d is

placed). Hence $x_n = x_m$ and $v = w$ as words, because the submonoid generated by x_0 and x_1 is free. \square

Let Γ_S be the induced subgraph of the Cayley graph Γ_F of F that contains all the vertices of from the set S (recall the definition of S in (6)). As a direct corollary of Lemma 1 and Lemma 2, we can describe explicitly the structure of Γ_S (see Figure 7, where solid edges are labelled by x_1 and dashed by x_0).

Proposition 6. *The structure of Γ_S is as follows*

- (a) Γ_S contains the infinite binary tree T corresponding to the free submonoid generated by x_0 and x_1 ;
- (b) for each $n \geq 2$ there is a binary tree T_n in Γ_S consisting of $n-2$ levels which grows from the vertex x_n and does not intersect anything else;
- (c) Each vertex x_nv of the boundary of T_n (i.e. v has length $n-2$) has two neighbors $x_nv x_1$ and $x_nv x_0$ outside T_n . The first one is the root of an infinite binary tree which does not intersect anything else. The second one coincides with the vertex $vx_0 x_1$ of the binary tree T .

Proposition 7. *The graph Γ_S is non-amenable.*

In order to prove this Proposition we will use equivalent to the amenability doubling condition (or Gromov doubling condition) [dlAGCS99].

Theorem A (Gromov's Doubling Condition). *Let X be a connected graph of bounded degree. Then X is non-amenable if and only if there is some $k \geq 1$ such that for any finite nonempty subset $S \subset V(X)$ we have*

$$|\mathcal{N}_k(S)| \geq 2|S|,$$

where $\mathcal{N}_k(S)$ is the set of all vertices v of X such that $d_X(v, S) \leq k$.

Proof of Proposition 7. In order to use the Theorem A it is enough to construct two injective maps $f, g : V(X) \rightarrow V(X)$ with distinct images, that do not move vertices farther than by distance k .

For any vertex x_nv in S put

$$f(x_nv) = x_nv x_1 x_0,$$

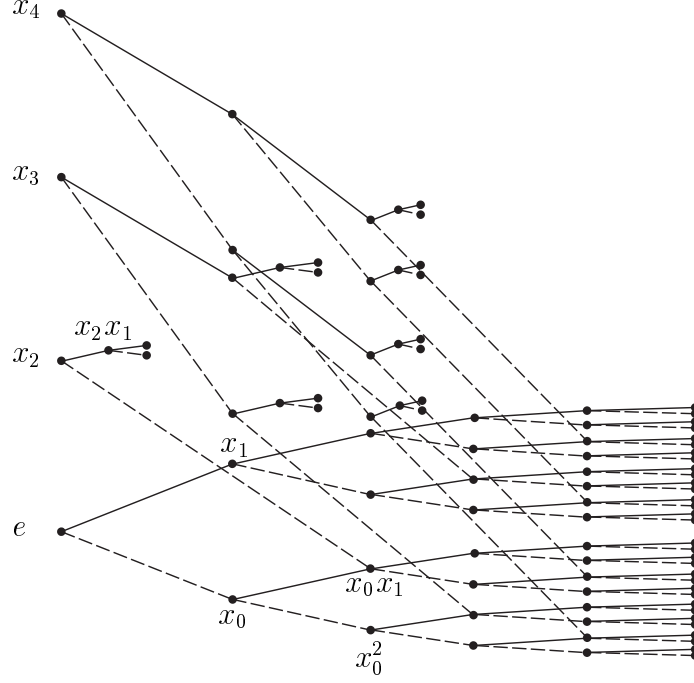


Figure 7: Induced subgraph Γ_S of the Cayley graph of F

$$g(x_nv) = x_nv x_1 x_1.$$

For any vertex x_nv of S we have $d(x_nv, f(x_nv)) = 2$ and $d(x_nv, g(x_nv)) = 2$, so the last condition of Theorem A is satisfied.

The relation $f(x_nv) = f(x_mw)$ implies $x_nv x_1 x_0 = x_mw x_1 x_0$ and $x_nv = x_mw$. Hence f is an injection. The same is true for g .

Now suppose $f(x_nv) = g(x_mw)$ or, equivalently,

$$x_nv x_1 x_0 = x_mw x_1 x_1 \quad (7)$$

The words $x_nv x_1$ and $x_mw x_1$ represent different vertices in Γ_S since otherwise we would get $x_0 = x_1$. According to Proposition 6 the equality (7) is possible only in case when $x_nv x_1$ is a vertex of the boundary of T_n and $x_mw x_1$ is a vertex of T . But by Proposition 6(c) in this case the vertex $x_nv x_1 x_0$ coincides with the vertex $v x_1 x_0 x_1$ of T which can not coincide with $x_mw x_1 x_1$. Indeed, otherwise we get

$$v x_1 x_0 = x_mw x_1.$$

Then vx_1 and x_mw must represent different vertices of Γ_S . According to Proposition 6 the last equality may occur only in case when vx_1 belongs to the boundary of tree T_r for some $r \geq 2$, which is not the case because $vx_1 \in T$. Therefore the equality (7) is never satisfied and the images of f and g are distinct.

Thus by Theorem A the graph Γ_S is non-amenable. \square

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